LIMIT OF FUNCTION

1. Limit of a function f(x) is said to exist as $x \to a$ when,

$$\underset{h\to 0^+}{\text{Limit}}\ f(a-h) = \underset{h\to 0^+}{\text{Limit}}\ f(a+h) = \text{some finite value M.}$$

(Left hand limit) (Right hand limit)

2. Indeterminant Forms:

$\frac{0}{0}, \frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, ∞^o, 0^o, and 1^∞.

3. Standard Limits:

$$\underset{x\to 0}{\text{Limit}} \frac{\sin x}{x} = \underset{x\to 0}{\text{Limit}} \frac{\tan x}{x} = \underset{x\to 0}{\text{Limit}} \frac{\tan^{-1}x}{x}$$

$$= \underset{x\to 0}{\text{Limit}} \frac{\sin^{-1}x}{x} = \underset{x\to 0}{\text{Limit}} \frac{e^x-1}{x} = \underset{x\to 0}{\text{Limit}} \frac{\ell n(1+x)}{x} = 1$$

$$\underset{x\to 0}{\text{Limit}}\ (1+x)^{1/x} = \underset{x\to \infty}{\text{Limit}} \left(1+\frac{1}{x}\right)^x = e, \quad \underset{x\to 0}{\text{Limit}} \frac{a^x-1}{x} = \log_e a,\ a > 0,$$

$$\underset{x\to a}{\text{Limit}} \frac{x^n-a^n}{x-a} = na^{n-1}.$$

4. Limits Using Expansion

(i) $a^x = 1 + \frac{x \ln a}{1!} + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + \ldots\ldots\ldots a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots \text{for } -1 < x \le 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(vii) for $|x| < 1$, $n \in R$ $(1 + x)^n$

$$= 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + ...\infty$$

5. Limits of form 1^∞, 0^0, ∞^0

Also for $(1)^\infty$ type of problems we can use following rules.

$\lim\limits_{x\to 0}(1 + x)^{1/x} = e$, $\lim\limits_{x\to a}[f(x)]^{g(x)}$,

where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a$ $= e^{\lim\limits_{x\to a}[f(x)-1]\,g(x)}$

6. Sandwich Theorem or Squeeze Play Theorem:

If $f(x) \le g(x) \le h(x)$ $\forall$ x & $\text{Limit}_{x\to a} f(x) = \ell = \text{Limit}_{x\to a} h(x)$ then $\text{Limit}_{x\to a} g(x) = \ell$.

METHOD OF DIFFERENTIATION

1. Differentiation of some elementary functions

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(a^x) = a^x \ell n\, a$

3. $\frac{d}{dx}(\ell n\,|x|) = \frac{1}{x}$

4. $\frac{d}{dx}(\log_a x) = \frac{1}{x\,\ell n\, a}$

5. $\frac{d}{dx}(\sin x) = \cos x$

6. $\frac{d}{dx}(\cos x) = -\sin x$

7. $\frac{d}{dx}(\sec x) = \sec x \tan x$

8. $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

9. $\frac{d}{dx}(\tan x) = \sec^2 x$

10. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

2. Basic Theorems

1. $\frac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$ **2.** $\frac{d}{dx}(k\,f(x)) = k\,\frac{d}{dx}f(x)$

3. $\frac{d}{dx}(f(x)\,.\,g(x)) = f(x)\,g'(x) + g(x)\,f'(x)$

4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\,f'(x) - f(x)g'(x)}{g^2(x)}$ **5.** $\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$

Derivative Of Inverse Trigonometric Functions.

$\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}$, $\frac{d\cos^{-1}x}{dx} = -\frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$.

$\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}$, $\frac{d\cot^{-1}x}{dx} = -\frac{1}{1+x^2}$ $(x \in R)$

$\frac{d\sec^{-1}x}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$, $\frac{d\,\mathrm{cosec}^{-1}x}{dx}$

$= -\frac{1}{|x|\sqrt{x^2-1}}$, for $x \in (-\infty, -1) \cup (1, \infty)$

3. Differentiation using substitution

Following substitutions are normally used to simplify these expression.

(i) $\sqrt{x^2+a^2}$ by substituting $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(ii) $\sqrt{a^2-x^2}$ by substituting $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(iii) $\sqrt{x^2-a^2}$ by substituting $x = a\sec\theta$, where $\theta \in [0, \pi]$, $\theta \ne \frac{\pi}{2}$

(iv) $\sqrt{\frac{x+a}{a-x}}$ by substituting $x = a\cos\theta$, where $\theta \in (0, \pi]$.

4. Parametric Differentiation

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

5. Derivative of one function with respect to another

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

6. If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h, l, m, n, u, v, w are differentiable

functions of x then $F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} +$

$$\begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

APPLICATION OF DERIVATIVES

1. Equation of tangent and normal

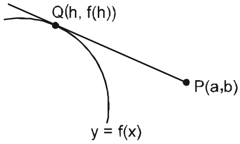
Tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$; when, $f'(x_1)$ is real.

And normal at (x_1, y_1) is $(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

2. Tangent from an external point

Given a point P(a, b) which does not lie on the curve y = f(x), then the equation of possible tangents to the curve y = f(x), passing through (a, b) can be found by solving for the point of contact Q.

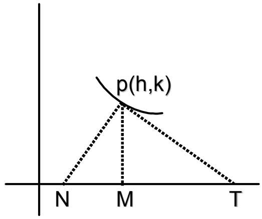
$$f'(h) = \frac{f(h) - b}{h - a}$$



Figure

And equation of tangent is $y - b = \frac{f(h) - b}{h - a}(x - a)$

3. Length of tangent, normal, subtangent, subnormal



(i) $PT = |k|\sqrt{1 + \frac{1}{m^2}}$ = Length of Tangent

(ii) $PN = |k|\sqrt{1 + m^2}$ = Length of Normal

(iii) $TM = \left|\frac{k}{m}\right|$ = Length of subtangent

(iv) $MN = |km|$ = Length of subnormal.

4. Angle between the curves

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves (as shown in figure).

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

5. Shortest distance between two curves

Shortest distance between two non-intersecting differentiable curves is always along their common normal.
(Wherever defined)

6. Rolle's Theorem :

If a function f defined on [a, b] is

(i) continuous on [a, b]

(ii) derivable on (a, b) and

(iii) f(a) = f(b),

then there exists at least one real number c between a and b (a < c < b) such that $f'(c) = 0$

7. Lagrange's Mean Value Theorem (LMVT) :

If a function f defined on [a, b] is

(i) continuous on [a, b] and (ii) derivable on (a, b)

then there exists at least one real numbers between a and b (a < c < b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

8. Useful Formulae of Mensuration to Remember :

1. Volume of a cuboid = ℓbh.
2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.
3. Volume of cube = a^3
4. Surface area of cube = $6a^2$
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.
6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height)
7. Curved surface area of a cylinder = $2\pi rh$.
8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.
9. Volume of a sphere = $\frac{4}{3}\pi r^3$.
10. Surface area of a sphere = $4\pi r^2$.
11. Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.
12. Volume of a prism = (area of the base) × (height).
13. Lateral surface area of a prism = (perimeter of the base) × (height).
14. Total surface area of a prism = (lateral surface area) + 2 (area of the base)
    (Note that lateral surfaces of a prism are all rectangle).
15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).
16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).
    (Note that slant surfaces of a pyramid are triangles).